Dominated assets, the expected utility maxim, and mean-variance portfolio selection

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ABSTRACT

Portfolio construction using both the expected utility maxim and mean variance selection is considered in the presence of a dominated asset. The analysis demonstrates, for selected case examples, that the expected utility maximizer will never hold the dominated asset long, while some portfolios along the mean variance efficient frontier contain long holdings of the dominated asset. It is argued that this result demonstrates another "weakness" of mean variance portfolio selection.

Keywords: Portfolio Choice, Dominated Assets, Expected Utility Maximization, Mean-Variance Portfolio Selection

INTRODUCTION*

The purpose of this paper is to consider optimal portfolio selection when a dominated asset is included in the menu of investment opportunities. Asset A dominates asset B if (1) the cash payments to A are at least as high as those to B and strictly greater than the payoff to B in at least one possible state outcome, and (2) if the current price of A is less than or equal to the price of B.

Of course, in highly developed capital markets with tight bid/offer spreads and ready information, one would not expect to observe dominating assets on an ongoing basis. Arbitrageurs would execute riskless arbitrage trades by shorting the dominated assets (asset B) and taking a long position in the dominating asset (asset A). This trading activity, when done in sufficient size, will drive the value of asset A up and the value of asset B down. Profitable arbitrage trades would continue to exist until the value of asset A was greater than the value of asset B.

However, in less than perfect markets, sufficient frictions could exist so as to make the arbitrage trade infeasible. In this case, the dominating/dominated asset relationship could persist. For example, if shorting under ideal conditions with full use of the proceeds is not available, the arbitrage trade could be difficult or impossible to execute[1]. Similarly, wide bid/offer spreads and brokerage transaction costs could also eliminate otherwise riskless arbitrage trades.

In this paper, portfolio selection will be studied assuming a dominated asset exists. The investor's portfolio decision will be considered under mean-variance portfolio selection (MV) compared with portfolio selection using the expected utility of terminal wealth maximization maxim (EU). It is well known that MV is a special case of EU if security returns are assumed to be normally distributed or if agents are assumed to possess quadratic utility functions, see the seminal works of Markowitz (1959) and Sharpe (1970). Furthermore, Merton (1969, 1971) has shown that MV is "approximately" correct in a multi-period, continuous time setting.

Considerable empirical research on MV over the past 30 years has consistently shown flaws in the MV model. As such, much theoretical work has been done to re-engineer the MV paradigm to make it more empirically reliable. See, for example, Fama and French (1992). While none of these extensions has been fully satisfying, analysts and practitioners continue to employ the MV apparatus even when neither of the two necessary conditions is likely to hold (i.e., normality or quadratic utility) and in the presence of conflicting empirical results.

Many financial economists would argue that the expected utility of terminal wealth maximization maxim is a more basic decision making criterion function than MV. However, because it is utility function dependent, it is less tractable than MV, especially when equilibrium and market clearing conditions are imposed on the model. For example, to impose equilibrium, the analyst would necessarily need to aggregate utility

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[1] Of course, an investor who is long the dominated asset could simply sell this asset and replace it with the dominating asset, thereby avoiding a need to execute a short sale.

functions of all market participants. Nonetheless, any more tractable portfolio selection procedure should be consistent with EU.

The aim of this paper will be to explore whether MV and EU make compatible portfolio selection decisions in the presence of a dominating asset. If not, another "weakness" in the MV method had been demonstrated.

In Section II, the general portfolio optimization problem under MV and EU will be presented. Section III provides an example, in the presence of a dominated asset, in which MV is inconsistent with EU. In particular, it is shown that the dominated asset is included in some portfolios along the mean-variance efficient portfolio frontier. As such, some investors would end up holding a long position in the dominated asset in their final portfolio. However, in a companion EU decision making approach, the dominated asset is not held long in the final portfolio.

The menu of assets in the investment opportunity set is provided using the time-state framework, making it easy to introduce dominating/dominated assets to the menu. A three asset, three state model is presented to highlight the inconsistency between MV and EU when a dominated asset is present.

Section IV is a brief summary.

EXPECTED UTILITY MAXIMIZATION AND MEAN-VARIANCE PORTFOLIO OPTIMIZATION

Employing the time-state preference model, the expected utility of terminal wealth maximizer would make optimal portfolio decisions as follows.

Let

N = # of securities in the investment opportunity set

M = # of outcome states; M is presumed to fully span the outcome space

C_{ij} = cashflow to security i if state j obtains

N_i = # of shares of security i purchased

P_i = price/share of security i at time 0

Π_j = probability that state j occurs

$\sum_{j=1}^{M} \Pi_j = 1$

W_o = initial wealth of the investor

$U(W)$ = utility of terminal wealth function that is monotonically increasing and concave

The EU decision rule has the investor maximize expected utility by choosing security investments subject to the initial wealth budget constraint.

$$\underset{N_i}{Max} \quad \sum_{j=1}^{M} \Pi_j \, U\left[\sum_{i=1}^{N} N_i C_{ij}\right]$$

$$s.t. \quad W_o = \sum_{i=1}^{N} N_i P_i$$

Forming the Lagrangian,

$$\underset{N_i}{Max} \quad \sum_{j=1}^{M} \Pi_j \; U\left[\sum_{i=1}^{N} N_i C_{ij}\right] + \lambda\left[W_0 - \sum_{i=1}^{N} N_i P_i\right]$$

The (N+1) first order conditions are:

$$\sum_{j=1}^{M} \Pi_j \; U'\left[\sum_{k=1}^{N} N_k^* C_{kj}\right] C_{ij} - \lambda^* P_i = 0 \quad for \quad i = 1, 2 \cdots N$$

$$W_0 - \sum_{k=1}^{N} N_k^* P_k = 0$$

In general, these first order conditions are (N+1) non-linear equations in the (N+1) unknowns $N_1^*, N_2^*, \cdots N_N^*, \lambda^*$.

The mean-variance portfolio optimization approach solves for the efficient portfolio frontier – the locus of portfolios in which, for any level of expected return, the variance of return is minimized. Individual investors would then choose from this set of frontier portfolios so as to maximize utility. Note that for any point on the efficient frontier, it is possible to exhibit a specific utility function for which the chosen point is the final optimal portfolio[2]. The derivation below will follow Merton (1972).

It is standard when performing mean-variance optimization to work with the proportion of total wealth held in each asset as opposed to the number of shares of each asset held.

Let

w_i = proportion of security i held in the portfolio

By definition

$$w_i = \frac{N_i P_i}{\sum_{k=1}^{N} N_k P_k} = \frac{N_i P_i}{W_0}$$

Define

σ_{ij} = covariance of return between security i and security j

$\sigma_{ii} = \sigma_i^2$ = variance of return on security i

μ_i = expected return on security i

For any expected return level Ø, MV will find the minimum variance portfolio to deliver the expected return Ø.

[2] This analysis has chosen not to include a riskless asset in the menu of investment opportunities.

$$\underset{w_i}{Min} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i \, w_j \sigma_{i\,j}$$

$$s.t. \sum_{i=1}^{N} w_i = 1$$

$$\sum_{i=1}^{N} w_i \; \mu_i = Ø$$

Forming the Lagrangian,

$$\underset{w_i}{Min} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i \; w_j \sigma_{i\,j} + \lambda_1\left[1-\sum_{i=1}^{N} w_i\right] + \lambda_2\left[Ø - \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{w}_{\mathrm{i}} \mu_1\right]$$

The (N+2) first order conditions are linear in $w_1^*, w_2^* \cdots w_N^*, \lambda_1^*, \lambda_2^*$:

$$\sum_{j=1}^{N} w_j \sigma_{i\,j} - \lambda_1 - \lambda_2 \mu_i = 0 \quad for \; i = 1, 2 \cdots N$$

$$1 - \sum_{i=1}^{N} w_i = 0$$

$$Ø - \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{w}_{\mathrm{i}} \mu_i = 0$$

Merton (1972) has shown that the solution to the first order conditions is a parabola in mean-variance space and a hyperbola in mean-standard deviation space.

MV AND EU INCONSISTENCY – AN EXAMPLE

Consider the following simple 3 asset, 3 state tableau:

State	Payoff to Asset #1	Payoff to Asset #2	Payoff to Asset #3	State Probability
1	$15	$15	$12	.33
2	20	20	9	.33
3	35	25	10	.34

If

P_1 = $18.65
P_2 = $18.65
P_3 = $ 9.84

it is clear that security #1 dominates security #2[3]. In anticipation of performing MV optimization, the rate of return and covariance (σ_{ij}) of return matrices are prepared.

R_{ij} = rate of return on security i if state j obtains

Rate of Return

State	Asset # 1	Asset # 2	Asset # 3
	-.20	-.20	.22
	.07	.07	-.09
	.88	.34	.02

For any security i, the expected rate of return, μ_i is

$$\mu_i = \sum_{j=1}^{3} \Pi_j R_{ij}$$

so that

$$\mu_1 = .2571$$

$$\mu_2 = .0750$$

$$\mu_3 = .0500$$

Covariance Matrix (σ_{ij})

Asset # 1	Asset # 2	Asset # 3
.2094	.0965	-.0242
.0965	.0481	-.0181
-.0242	-.0181	.0159

where

$$\sigma_{ij} = \sum_{k=1}^{3} \Pi_k (R_{ik} - \mu_i)(R_{jk} - \mu_j)$$

Now, given this financial data for the three assets in the investment opportunity set, MATLAB is used to solve the (N+2) MV first order conditions for a given Ø. To trace out the efficient portfolio frontier, the first order conditions are solved for different values of Ø. That is, for any Ø value, MATLAB solves the MV constrained minimization problem to select optimal portfolio weights w_1^*, w_2^*, w_3^*. Using these three portfolio weights in conjunction with the covariance matrix, the corresponding portfolio standard deviation is computed.

The relevant question becomes: Are there portfolios along the efficient frontier in which $w_2^* > 0$? Stated differently: Is a dominated asset held long in any mean-variance efficient portfolio? If so, there is the likelihood that some investors would wish to hold the dominated asset long in their final portfolio.

For the 3 asset scenario that was constructed, the answer to these questions is YES.

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[3] Payoffs to the two securities are identical in states #1 and #2, but the payoff is higher to asset #1 than asset #2 in state #3. To avoid dominance $P_1 > P_2$, but by construction $P_1 = P_2$.

MATLAB runs provide $(Ø, \sigma)$ pairs which trace the efficient frontier along with the related portfolio weights w_1^*, w_2^*, and w_3^* for each efficient portfolio[4]. For the numerical example, along this efficient frontier, Ø ranges between 5.7% and 25.7% while σ varies between 6.6% and 45.8%. It is also noted that in the Ø range between 5.8% and 10.0%, $w_2^* > 0$. So for low expected return (and low variance) efficient portfolios, the dominated asset (asset #2) comes into the portfolio with a <u>positive</u> portfolio weight meaning that it is to be held long in that particular portfolio. This counterintuitive result may arise from the fact that asset #2 is significantly negatively correlated with asset #3, $\rho_{23} = -0.65$, and that MV includes asset #2 in these portfolios to take advantage of its variance reduction (diversification) properties. The dominant asset #1 is also negatively correlated with asset #3, $\rho_{13} = -0.42$, and highly correlated with asset #2, $\rho_{12} = 0.96$. In this example, when Ø>10%, apparently the benefits of diversification provided by asset #2 are outweighed by its low (relative to the dominate asset #1) expected return contribution so that $w_2^* = 0$[5].

An important point from this MV analysis is that there exist optimal portfolios which hold the dominated asset long. Below it is shown that the EU analysis (again via example) leads to optimal portfolios in which the dominated asset is <u>not</u> held long. This result will illustrate the inconsistency between the MV and EU models.

For the EU illustration, consider an investor with quadratic utility of terminal wealth facing the same 3 asset investment opportunity set as above. The utility function is written as:

$$\begin{aligned} U(W) &= W - \frac{b}{2}W^2 \qquad b > 0,\ 0 \le W \le \frac{1}{b} \\ &= \frac{1}{2}b \qquad\qquad W > \frac{1}{b} \end{aligned}$$

For the numerical example, choose the parameter b so that, given the asset choices, U (W) is increasing so as to avoid satiation. Since $U'(W) = 1 - bW$, first order conditions (3) and (4) become

$$\sum_{j=1}^{M} \Pi_j \left[1 - b\sum_{k=1}^{N} N_k^* C_{kj}\right] C_{ij} - \lambda^* P_i = 0 \qquad for\ i = 1, 2, 3 \tag{3'}$$

$$W_0 - \sum_{k=1}^{N} N_k^* P_k = 0 \tag{4'}$$

Choosing b = 0.01 and W_0 = \$50, the solution to the 4 equation system (3') and (4'), which is linear in { N_k^* } and λ^* is:

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[4] The complete set of MATLAB data is available on request.

[5] Note that the particular version of MATLAB used constrained all portfolio weights to be non-negative. If this were not the case, possible shorting $(w_2^* < 0)$ of asset #2 would be expected.

$N_1^* = 6.17$

$N_2^* = -8.25$

$N_3^* = 9.02$

$\lambda^* = 0.27$

Note that N_2^*, the number of shares of the dominated asset to be held, is negative. That is, in the optimal portfolio for this quadratic utility investor, the dominated asset is to be shorted. This result is in stark contrast to the MV solution in which the dominated asset can be held long.

To demonstrate that the result is robust for a wider range of quadratic utility functions, a sensitivity analysis on b was performed. Appendix A shows that N_2^* will be negative for a range of parameter b choices that insure non-satiation.

SUMMARY

In this paper it was shown, by way of example, if investors have quadratic utility functions and are expected utility of terminal wealth maximizers, they will never hold a dominated asset in their final portfolio. It was also shown that there exist mean-variance efficient portfolios that contain long holdings of the dominated asset. Therefore, under the MV approach, it is possible that some (quadratic utility) investors choose portfolios containing long positions in the dominated asset. Yet the direct EU calculations demonstrate that this is never the case for such investors.

While this is a very special case example of the inconsistency of MV and EU, it does highlight another area of "weakness" associated with mean-variance analysis.

Areas of further research could include expanding consideration to a more comprehensive investment opportunity set and consideration of other utility function classes in the EU analysis. Further, it would be productive to study why MV chooses long holdings of the dominated asset for some portfolios along the efficient frontier. For example, does the dominated asset provide diversification benefits that outweigh its lower (than the dominating asset) expected returns? Further, what are the characteristics of the regions on the efficient frontier in which the dominated asset is held long?

APPENDIX A: EU SOLUTION WITH QUADRATIC UTILITY – SENSITIVITY TO PARAMETER b

b	**N_2***
.001	-199.31
.002	-93.17
.003	-57.78
.004	-40.09
.005	-29.48
.006	-22.40
.007	-17.35
.008	-13.56
.009	-10.61
.010	-8.25
.011	-6.32
.012	-4.71
.013	-3.35
.014	-2.19
.015	-1.18
.016	-0.29

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