From description to prescription: a proposed theory of teaching coherent with the Pirie-Kieren Model for the dynamical growth of mathematical understanding

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ABSTRACT

Davis and Sumara (2010) point out that learning is complex, which is evident to anyone who has strived to support students' learning. However, it is perhaps less apparent that complexity theory can be useful for its power both to unify various theories of learning and to provide a foundation to foster understanding among learners and teachers. The attributes of a complex system—emergent, dynamic, co-adaptive, nonlinear, recursive, nested processes—are readily observable in the learning and learning-teaching environments. The benefits of seeing such environments through the lens of complexity theory are unification, clarification, and a suggested direction for progress. In this paper, I propose a theory of teaching that is coherent with the Pirie-Kieren Model for the Dynamical Growth of Mathematical Understanding, and situated within complexity theory as a superordinate framework.

Keywords: Description, prescription, coherent

TRANSFER

Spiro and De Schryver (2009) discuss this problem by contrasting Well-Structured Domains (WSDs) and Ill-Structured Domains (ISDs). WSDs are generally closer (and sometimes identical) to the contexts in which knowledge and skills are learned; they also tend to be more closely related temporally to the learning of skills. ISDs lack most/all of these attributes and instead are viewed as being "indeterminate, inexact, noncodifiable, nonalgorithmic, nonroutinizable, imperfectly predictable, nondecomposable into additive elements, and, in various ways, disorderly" (Spiro & De Schryver, 2009, p. 107). In addition, ISDs tend to also be temporally further into the future. Thus, although far transfer is identified as a major goal of formal education, there is little evidence that this type of transfer occurs. This lack of scientific evidence for far transfer has been used by critics to argue that direct instruction is superior to constructivist-based instruction (Kirschner, 2009; Mayer, 2009; Sweller, 2009). Yet, Spiro and De Schryver (2009) point out that scientific evidence of far transfer is impossible, given the attributes of situations in which far transfer may occur, especially temporally (it might be years before an ISD requires far transfer). Clearly, no empirical research structure would be possible. Barnett and Ceci (2002) have offered a taxonomy for far transfer (see Table 1) that concisely summarizes the continuum of near versus far transfer; they also point out that "Children ... transferred when they developed a deep, rather than surface, understanding" (p. 616). Therefore, since transfer is central to learning, any theory of learning or teaching must address the need for deep learning.

Spiro, Coulson, Feltovich, and Anderson (1994) have advanced a Cognitive Flexibility Theory (CFT) specifically targeted for far transfer. Their theory is heavily structured to avoid what they call learning misconceptions that students acquire from exposure to strategies aimed at near transfer. For example, Spiro et al. list as necessary elements: avoidance of oversimplification and over regularization; multiple representations/schemas; centrality of cases (bottom-up vs. top-down analysis); conceptual knowledge as knowledge in use; schema assembly (assemble knowledge from different conceptual and precedent cases, rather than retrieval and accretion of existing schema; noncompartmentalization of concepts and cases (multiple interconnectedness); active participation, tutorial guidance, and adjunct support for the management of complexity. CFT recognizes that "the learner must attain a deeper understanding of content material, reason with it, and apply it flexibly in diverse contexts" (Spiro et al., 1994, p. 2). Spiro et al. apply CFT to situations in medical training, requiring transfer in ISDs, with some temporal factor (near or far).

COMPLEXITY THEORY AND EPISTEMOLOGIES OF LEARNING

In this section, instead of debating the various theories' attributes of various theories—since as Davis (1996) points out that there are hundreds and possibly thousands of theories of learning—I discuss various epistemologies of learning and show how complexity theory can serve as a unifying concept.

Constructivism is based on the theories of Piaget and represents "an effort to construe personal learning through the metaphor of emergent biological forms, the structures of which are conditioned but never determined by their contexts—hence his use of terms such as 'assimilation' and 'accommodation'" (Davis & Sumara, 2002, pp. 411-412). For Piaget, learning was an individual but not isolated activity in which "the individual knower was engaged in the unrelenting project of assembling a coherent interpretive system, constantly updating and revising explanations and expectations to account for new experiences" (Davis & Sumara, 2002, p. 413). This has strong echoes of Kelly's Personal Construct Theory, whereby individuals are constantly anticipating and predicting based on their current construct system (Hogan, Johnson, & Briggs, 1997), revising or rejecting personal constructs when experience causes cognitive dissonance. The UK Council for Psychotherapy classifies Personal Construct Theory as Experiential Constructivism. Interactions, including social, are essential under radical constructivism, but are seen as context rather than primary. Von Glasersfeld (1995) identifies constructivism as adaptation, with the goal of a coherent, conceptual organization of the world as experienced (p. 7).

Social constructivism foregrounds social interactions as the drivers of learning. Based on Vygotsky's notions of interpersonal preceding intrapersonal, cognition is diffuse, distributed, and collective (Davis & Sumara, 2002, p. 414). Gergen (2005) points out that language plays a very significant role in social constructivism, as a means of communicating meaning, where meaning is also context dependent. Vygotsky's work provides some informative concepts for teachers, such as the zone of proximal development (ZPD), which is the difference between what a learner can accomplish with the assistance of knowledgeable others (teachers, parents, other students) and what the learner can accomplish unaided (Hughes, 2014). The processes inside the ZPD are typically called scaffolding (Hughes, 2014) and consist not of telling the student an answer but rather asking questions, suggesting directions, directing students to other resources, and providing encouragement.

Sociocultural theories are related to Vygotsky's metaphor of shared labour. Davis and Sumara (2002) identify classroom-related facets of this theory, such as emphasis on group processing and the justification of positions within disciplines. There is a relationship here to situated cognition, with its concern for the processes by which individuals enter established communities of practice (Davis & Sumara, 2002).

Cobb (2005) identifies a key difference among the aforementioned positions as the unit of analysis, with radical constructivists' unit being the individual, and sociocultural and social constructivists' unit being the individual-in-social-action; however, Cobb acknowledges many similarities across the positions. Bauersfeld (2005), in arguing for "mathematizing as a social practice," also recognizes the multiplicity of positions within a mathematics classroom, sometimes competing but often complementary. Many of the complementarities reflect complexity-based concepts: emergence, biology-based metaphors for learning, dynamic, nonlinear, and self-similarity (Davis & Simmt, 2003). Thus, a complexity-based theory of learning subsumes many of the key features of all of the theories of learning described above. This position is especially useful when discussing theories of learning mathematics, such as the Pirie-Kieren model.

PIRIE-KIEREN MODEL FOR DYNAMICAL GROWTH OF MATHEMATICAL UNDERSTANDING

The Pirie-Kieren model for the dynamical growth of mathematical understanding is a constructivist model (Pirie & Kieren, 1992) consistent with, and representative of, a complexity orientation for learning. Pirie and Kieren (1994) formulated a model for mathematical understanding that is coherent with complexity theory, in that it is nonlinear, dynamic, active, and recursive. A representation of the model is shown in Figure 1.

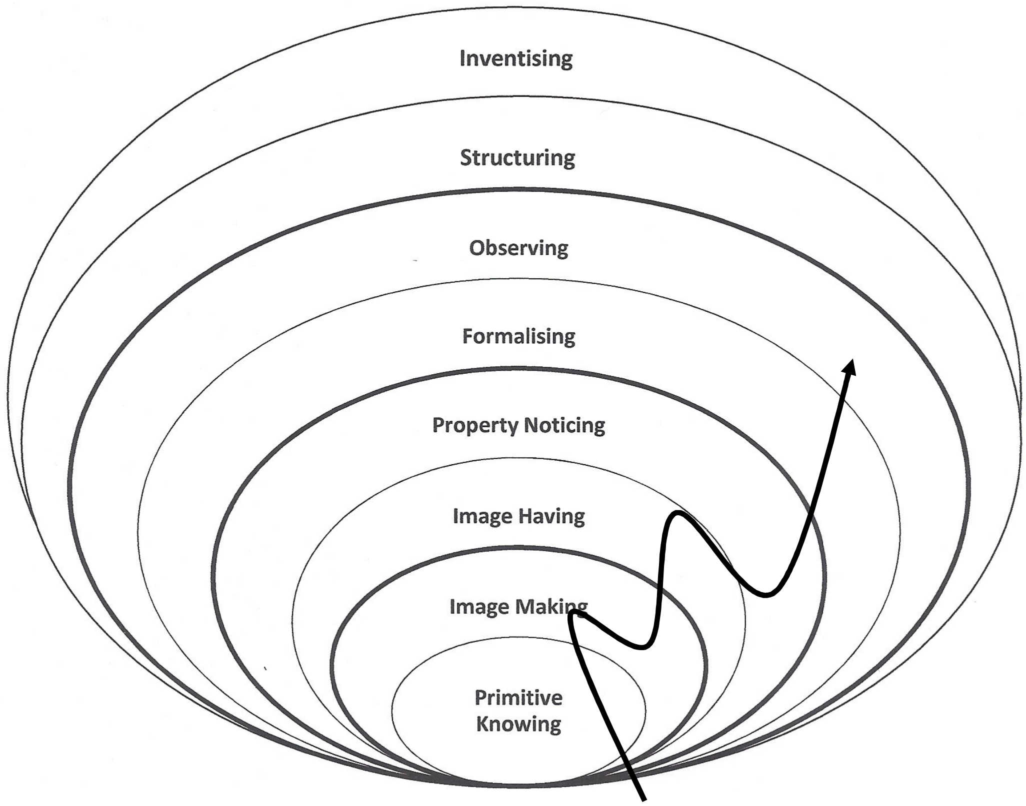

Figure 1. The Pirie-Kieren Model for the dynamical growth of mathematical understanding.

Primitive Knowing represents the learner's initial knowledge about the topic. The use of the word primitive does not imply rudimentary or low-level knowledge; simply, this is the learner's knowledge state prior to engaging. In the next level, *Image Making*, image means any representation, including mental, visual, pictorial, and so on. This is consistent with schema construction and adaptation. Schema are interconnected mental representations of prior knowledge, often compared to mental mind maps (Irvine, 2016; Widmayer, n.d.). Derry (1996) suggests that schema represent the big ideas fundamental to understanding. Constructivist theory (e.g., Fosnot, 2005) claims that learning occurs when students modify or build onto their existing schema for a particular topic. When considering a cognitive taxonomy such as Marzano's New Taxonomy (MNT), this level is analogous to a taxonomic-level Comprehension such as Symbolizing (Marzano & Kendall, 2001, 2007) whereby the student constructs mental or physical images of concepts. *Image Having* represents a level where actual image making is no longer necessary; the learner can use the image without resorting to image making or specific examples.

At the *Property Noticing* stage, the learner is able to construct context-specific properties by combining attributes of images. In a cognitive taxonomy such as MNT, this represents a taxonomic-level of Comprehension such as Integrating (Marzano & Kendall, 2001, 2007) in which students identify essential and non-essential characteristics of concepts. At *Formalising*, the learner abstracts a common attribute or method from the properties noticed in the previous stage. This level is analogous to Analysis-Matching level in a taxonomy (Marzano & Kendall, 2001, 2007). *Observing*, a level equivalent to taxonomic Analysis-Classifying (Marzano & Kendall, 2001, 2007), allows the learner to express coordinated formalizing as theorems. At the *Structuring* level, the learner collects appropriate theorems to form a coherent theory, the taxonomic Analysis-Generalizing (Marzano & Kendall, 2001, 2007). Finally, *Inventising* involves generating new questions based on a full, rich understanding of the topic, by breaking away from the preconceptions that enabled learners to reach this outer level. This is equivalent to the Analysis-Specifying or Predicting level in Marzano's taxonomy (Marzano & Kendall, 2001, 2007). In MNT, all Analysis levels are at a higher level than Comprehension levels.

There are three important features of the model. *Folding Back*, a dynamic, recursive process, involves revisiting previous levels to build understanding and allow the resolution of problems or questions that have occurred at a more outer level. Folding back is critical to building understanding. Because learners engaging in folding back return to the previous level but retain all the newer understanding that they have developed, Pirie and Kieren (1992) refer to this richer understanding as *Thickening*. Learners' knowledge is thus "thicker" or richer, as they return to previous levels and reconstruct their understanding using this new knowledge. This is an important feature of this model, and is at the heart of enriching learners' understanding. The third feature of the model is *"Don't Need" Boundaries*. Once beyond such boundaries, learners do not require a return to a specific prior level in order to proceed; the growth of "Don't Need" Boundaries provides a benchmark of the learners' growth in understanding. In Figure 1, "Don't Need" boundaries are identified with heavier lines. It is critical to recognize that learners do not proceed linearly towards the outer levels. Through folding back, learners proceed nonlinearly towards deeper understanding, which is often represented on the model by a "wandering" line that revisits earlier levels, advances, revisits, and so on, in an irregular, nonlinear, recursive manner.

Pirie and Kieren (1994) point out the interconnectedness of mathematics, with a process that I call *chaining*. They provide the example of a student whose understanding of fractions (at whatever level, probably imperfect, that such understanding exists) becomes the learner's primitive knowledge for understanding decimals. It is not possible to silo a topic to the exclusion of other related topics and concepts. This is a marvelous illustration of the complexity of mathematics and of learning—chaining is not meant to imply linearity, as the interrelated topics of mathematics form a rich, interconnected, multilayered structure that grows with the learner's growth of understanding. Figure 2 illustrates an (incomplete) example of chaining. The illustration is incomplete because learners will continue to grow and build on their current level of knowledge.

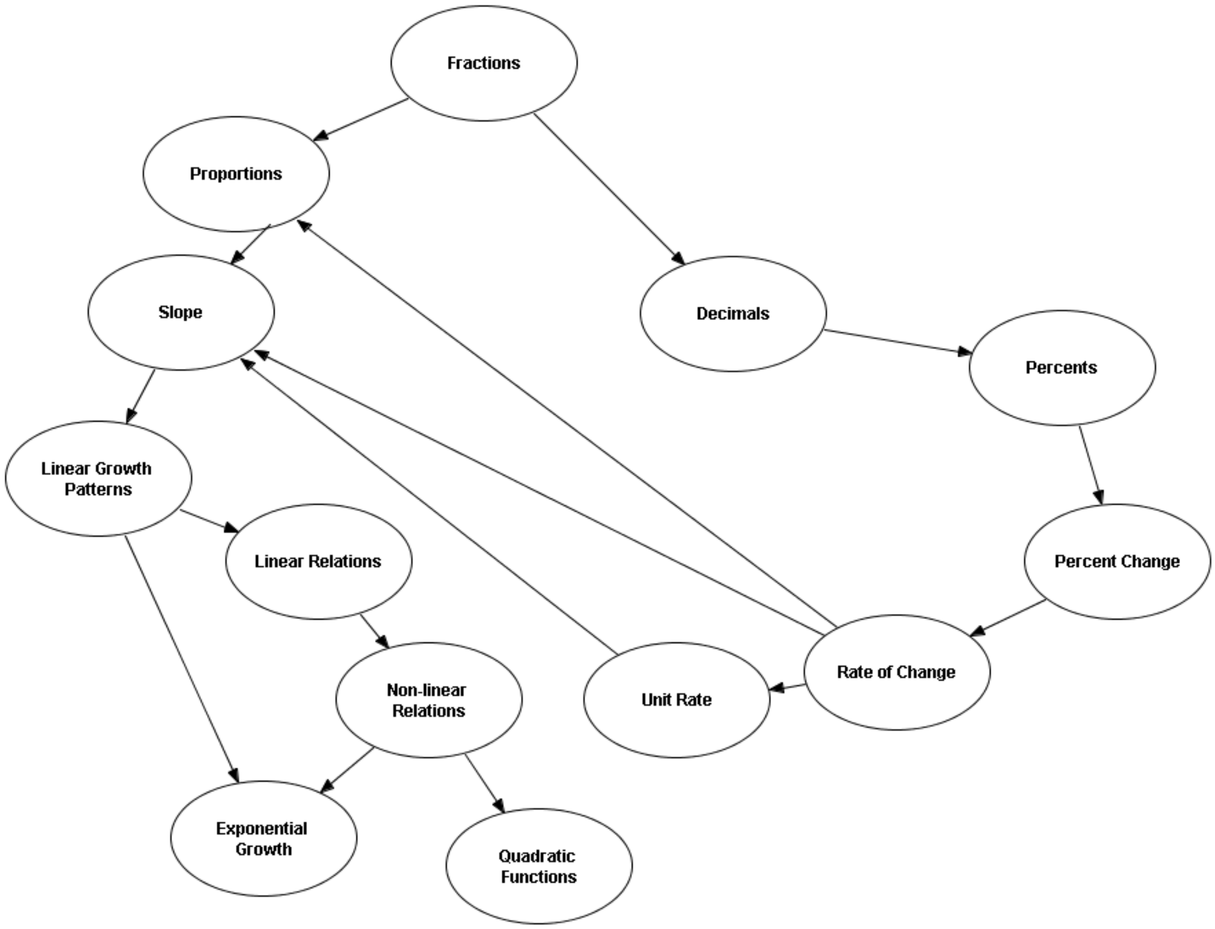

Figure 2. An (incomplete) example of chaining.

One of the strengths of the Pirie-Kieren model is that it can reflect not only individual learning but also pairs or group learning (Martin, Towers, & Pirie, 2006; Towers & Davis, 2002); with workplace training (Martin & LaCroix, 2008); with teacher candidates (Slaten, 2007); and with classroom teachers (Droujkova, Berenson, Slaten, & Tombes, 2005). Towers and Davis (2002) conducted an especially interesting study in which they produced folding back diagrams for two individual learners working as a pair, identifying points of convergence and divergence of the students' thinking in response to teacher prompts and questions.

A THEORY OF TEACHING COHERENT WITH THE PIRIE-KIEREN MODEL FOR THE DYNAMICAL GROWTH OF MATHEMATICAL UNDERSTANDING

While theories of learning such as Pirie-Kieren inform our thinking concerning what happens to the learner and engenders growth of understanding, they are descriptive and do not provide guidance for teachers as to what should be done to foster learners' growth. However, descriptive theories of learning may provide minimal guidance with respect to teaching. For example, Davis and Sumara (2002) indicate that teachers subscribing to a "constructivist" teaching orientation may enact a trivialized version of constructivist instruction, such as assuming that providing manipulatives is sufficient. It can also lead to absurdities such as “don’t tell the students anything” or (my personal favourite) “since teachers and students learn together, it is an advantage for teachers to have poor content knowledge, so they can grow together” (Davis & Sumara, 2002). Such statements represent a misunderstanding of the concept of learning together. While students and teachers alike learn, each is learning different things: Students are learning mathematical content and concepts and principles; teachers are learning about their students, and in addition may also learn different lenses through which to view particular mathematical content. However, students and teachers are not learning identical content nor in identical ways.

Attempting to teach with a constructivist view of learning may result in broader misconceptions. For example, a school district interpreted the work of Cathy Fosnot (Fosnot & Dolk, 2001) to mean that teachers were not to give students learning goals. Rather, the students were to “investigate” and somehow mathematical understanding would occur. Instead of being provided with a learning goal such as “Investigate whether there is a relationship between surface areas and volumes of cylinders,” students were provided various cylinders and teachers watched, without interacting, as the students floundered through the class. This has been a strong criticism of constructivist-based teaching (Kirschner, 2009; Mayer, 2009; Rosenshine, 2009; Sweller, 2009), where advocates of direct instruction define constructivist teaching as discovery inquiry, with no or minimal teacher guidance or intervention. It is clear that many teachers are unsure of what steps to take to foster student learning. Even when supported by professional learning, teachers either reject a constructivist-based approach or are unable to implement it effectively.

There are two contrasting reasons for this situation. In secondary school, teachers frequently reject a “new” or different approach based on their own significant content knowledge and continue to teach the way they were taught, usually in a transmission or instrumental mode. They often argue that this approach is best for their students since this is the dominant approach in postsecondary institutions, and the significant number of studies that indicate the lecture method is an inefficient and ineffective method of knowledge construction does little to sway their resolve. Alternatively, teachers in many Kindergarten through Grade 8 classrooms are unable to implement an effective constructivist based pedagogy because of their weak content knowledge, both mathematical and knowledge of the processes and practices that instigate mathematics learning. Ball and Bass (2003) refer to this as pedagogical content knowledge, or sometimes as content knowledge for teaching mathematics (CKTM). In spite of their best intentions, these teachers are unable to respond meaningfully to student endeavours, questions, or hypotheses, such as student-generated algorithms, conjectures, misconceptions, and lateral or nonlinear thinking. For many of these teachers, the mathematics curriculum is the textbook, which is dealt with in a linear, often mind-numbing manner. Without any clear indication of what they should do, even teachers who attempt to break this cycle of direct instruction either flounder, give up the approach, or resort to less than optimal instructional engagements.

Evidence of this failure is found in Ontario’s Education Quality and Accountability Office (EQAO) standardized tests; over the past 10 years, EQAO Grade 6 mathematics results have fluctuated between a mere 58% to 61% of students achieving Level 3 or Level 4 standards that meet or exceed provincial expectations, respectively (EQAO, 2012). While the EQAO assessments are certainly imperfect and subject to numerous criticisms, the failure to increase meaningful understanding among so many students despite high levels of financial and professional learning supports is a damning indictment of the manner in which “constructivist based teaching" has been implemented. Simply put, teachers lack clear, research-supported direction as to what they should be doing. What is needed is a theory of pedagogy that is coherent with theories of learning, such as the Pirie-Kieren model. Davis, Sumara, and Luce-Kapler (2008) point out that there is no causal link between teacher actions and student learning; what is required is that the teacher provides situations and frameworks that will support learners in deepening their own understanding (here I use "learners" collectively to identify individuals or groups of students progressing towards better conceptual understanding, although each possesses a different lens).

Various authors have identified aspects of teaching, often through metaphor: Davis (1994, 1997) focuses on teachers' "listening for differences" or hermeneutic listening (discussed later in this paper); Sfard (2001) identifies communication as paramount; Martin et al. (2006) uphold the importance of improvisation and students' participation in learning; Davis and Renert (2009) highlight shared participation; Warner and Schorr (2004) describe the value of student-to-student questioning; and Towers (2010) discusses teaching through inquiry. Still, while all of the latter studies identify important facets of teaching, none provide a comprehensive description of what teachers are expected to do to engender increased student understanding.

In this section, I propose a theory of teaching that is coherent with a constructivist/ complexity theory of learning (such as Pirie-Kieren). Anyone who has been in a classroom recognizes that teaching is a complex activity, particularly when meeting all the criteria for complexity outlined previously. It is dynamic, nonlinear, emergent, interactive, and frequently surprising. I use the metaphor of *nested systems*, like the Pirie-Kieren model, although Davis and Sumara's (2012) metaphor of decentralized networks is perhaps more appropriate, especially for the teaching model.

While there currently are numerous theories of teaching, the majority represent teaching as a linear (sometimes cyclical) process. Simon (1995), for example, identifies a number of important considerations for teachers to address, however he also presents a linear model for teaching, followed by a matrix model, both with considerable feedback loops. Similarly, Berenson, Mojica, Wilson, Lambertus, and Smith (2007) describe a teaching protocol based on students developing mathematical models, but in a linear progression. The concern here is that any linear or nearly linear model of teaching is a mismatch with a theory of learning that is complex (i.e., dynamic, nonlinear, recursive, and emergent).

Figure 3 illustrates a theory of teaching that is coherent with a constructivist/complexity-based theory of learning, such as the Pirie-Kieren model, using a similar metaphoric representation to emphasize the coherence. The components of the teaching theory are described below, followed by a discussion of the "pre-activities" in which the teacher engages.

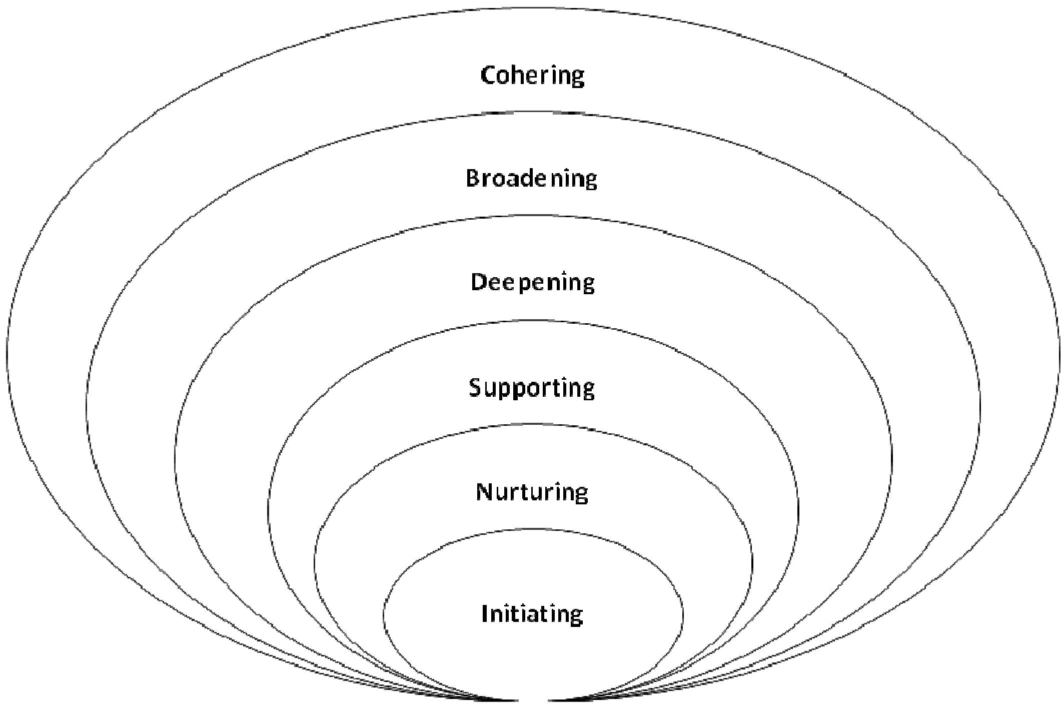

Figure 3. A theory of teaching coherent with a constructivist/complexity-based theory of learning.

Initiating is the initial teacher task, but not the initial teacher reflection. Initiating is the situation, problem, or inquiry topic that the teacher has identified as most appropriate for his or her students, for a particular concept. Once students are engaged in the initiating task, the teacher's role involves *Nurturing*. Nurturing refers to nurturing of the mind; it involves prompting, probing, questioning, seeking clarification, and identifying misconceptions. Nurturing revolves around what Davis (1995, 1996, 1997) calls *hermeneutic listening*. Davis (1997) describes hermeneutic listening as

> intended to reflect the negotiated and participatory nature of this manner of interacting with learners. ... This sort of listening is an imaginative participation in the formation and the transformation of experience. Hermeneutic listening demands the willingness to interrogate the taken for granted and the prejudices that frame our perceptions and actions. (pp. 369-370)

Both the learner and the teacher will engage in "folding back" actions during this phase, with repeated references to the initiating task, as both seek clarity and seek to uncover the current level of student understanding. The learner's need to fold back will determine when the teacher folds back. During this phase, concepts or skills that may be ancillary to the learner's understanding of the topic may be uncovered. This leads to the level of *Supporting*, whereby the teacher provides learning activities to assist students in moving forward with the main topic.

Once these ancillary activities are completed to the student's and teacher's satisfaction, folding back to the nurturing level will occur, as the student then moves on with understanding their principal concept. As a student's understanding grows, the teacher's task becomes structuring activities that promote *Deepening* of understanding. This will generally require folding back to the levels of supporting and nurturing. Deep learning is critical for enduring understanding (Irvine, 2016). Deepening will typically involve new or more in-depth subtasks for the learner, at the appropriate level and time. Similarly, the level of *Broadening* asks the teacher to provide tasks that will increase the breadth of students' understanding of the concept, making connections to other topics, the real world, possibly similar but also possibly dissimilar applications, and so on. This level addresses *transfer*, a critically important concept. As Martinez (2010) notes, "Transfer is so important that it arguably is the ultimate goal of education" (p. 111). Similarly, Perkins and Salomon (1988) identify transfer as "integral to our expectations and aspirations for education" (p. 22); they argue that knowledge and skills acquired in formal schooling are generally inert, and neither useful nor available for transfer. In particular, studies have shown that transfer is more likely to occur in situations of near transfer, and much less likely to occur for far transfer (Barnett & Ceci, 2002). Activities in Broadening will typically involve near transfer, although instances of far transfer could be undertaken depending on the student's readiness and the teacher's ability to recognize the student's needs. Finally, the teacher moves to the *Cohering* stage. Here the teacher attempts to move the student's knowledge into the realm of taken as shared, while recognizing that each student's understanding will differ in some respects to the teacher's as well as to other students' understanding. This stage is important for two reasons: First, logistically the teacher needs to deal with a class of students whose knowledge has some level of consistency; second, students must be able to deal with and communicate about concepts in a way that allows large-group discussion on somewhat common ground.

With every instance of folding back, the teacher's aim is to thicken students' understanding. Metaphorically, the teacher's theory of teaching is overlaid on students' respective theory of learning, in a complex dance of actions by both parties aimed at increasing student understanding.

WHAT HAPPENS BEFORE THE INITIATING TASK IS CHOSEN?

The initiating task serves several purposes. First, it activates students' prior knowledge, which entails activation of student schema called cognitive fields (Derry, 1996). Cognitive fields are a distributed pattern of memory activation that makes memory objects more available than others, based on a (initiating) event (Derry, 1996). The initiating event also serves to motivate students to engage in the task. Often this mandates that the initiating event be grounded in real life, particularly the students' real lives (Irvine, 2015). Motivation has been linked demonstrably to mathematics achievement, in a reciprocal relationship. This means that increased motivation correlates with increased achievement, and increased achievement correlates with increased motivation (Irvine, under review). Finally, initiating tasks may cause *cognitive dissonance*, which occurs when a situation conflicts with a student's pre-existing schema, causing the student to interrogate both the new situation and the pre-existing beliefs (Widmayer, n.d.). Prior to identifying an initiating task, the teacher must do a significant amount of "homework." Figure 4 outlines the preparatory activities (note here also that folding back will occur as the teacher identifies and rejects possible pathways and alternatives).

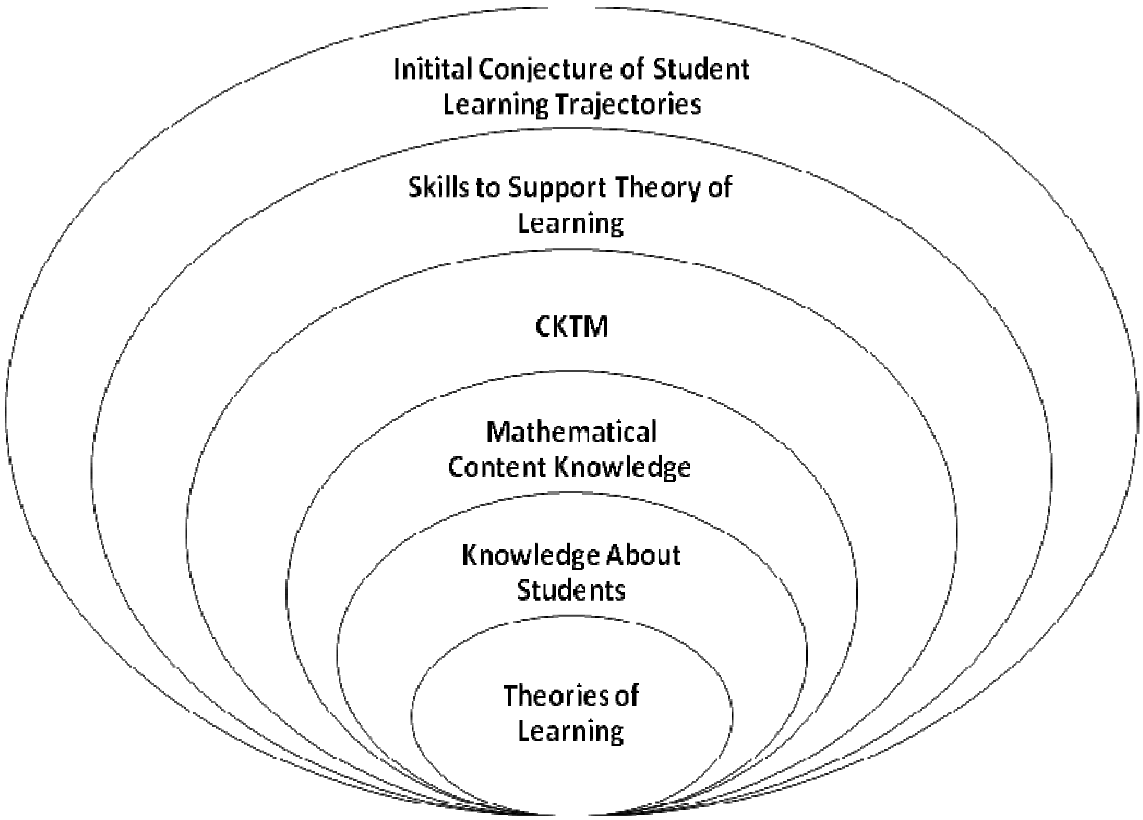

Figure 4. Teacher knowledge and skills for a theory of teaching coherent with constructivist/ complexity-based theory of learning: Requirements before choosing an initiating task.

The process begins with identifying an appropriate *Theory of Learning*, in this case the Pirie-Kieren theory. The second level involves the teacher's *Knowledge About Students*—their attitudes, interests, readiness, preferred learning styles, and other relevant background. Next the teacher must have a level of *Mathematical Content Knowledge* with respect to the topic under study. This level of knowledge need not be exhaustive, but must be sufficient to enable the teacher to support students' learning, recognize alternative or diverse pathways, and allow students to engage with the concept in diverse ways. At the next level the teacher must have the required *Content Knowledge for Teaching Mathematics* (CKTM). Ball and Bass (2003) posit that CKTM is qualitatively different from mathematics content knowledge; it involves knowing not only the mathematics content but also a variety of ways to address that content knowledge. This enables the teacher to respond appropriately, fully, and deeply to student-generated changes to the initial anticipated learning trajectory. Then, the teacher must possess the necessary *Skills to Support the Theory of Learning*, which include: hermeneutic listening, reflective questioning, skillful probing and clarifying, and the ability to urge and encourage students to take "the path less travelled." The teacher also needs to be skilled and knowledgeable in activities involving framing, for example: Bansho (Literacy and Numeracy Secretariat, 2010), Math Congress (Fosnot, 2005), and Math Forum (Literacy and Numeracy Secretariat, 2011), which are all consolidation related structures. Knowledge is also required in relation to frames for learning that are coherent with the theory of learning, such as concept attainment, placemat, anticipation guides, and other appropriate instructional strategies. Finally, the teacher should have dynamic classroom skills, because teaching in this way tends to be very active, noisy, and energized. The last stage is *Initial Conjecture of Student Learning Strategies*; Simon (1995) emphasizes that conjecturing initial student learning trajectories is particularly important. Despite recognizing that these learning trajectories will perforce change, such initial conjectures will shape the initiating task that is selected.

CONCLUSION

It is critical that any theory of teaching be coherent with the theory of learning under which students are engaged. Implementing the theory of teaching outlined above will require a significant commitment to job-embedded professional development. This will be necessary for teachers currently in the classroom, as well as requiring modifications to pre-service teacher education programs. Pathways will also be needed to encourage teachers of senior mathematics courses to contemplate and hopefully implement this theory. Their relentless focus on mathematics content will be difficult to overcome.

At the elementary level a significant impediment is teacher self-efficacy, which Ross (2009) has shown to be stable and difficult to change, as well as teacher attitudes towards mathematics. Job-embedded professional learning at this level will also need to address mathematical content knowledge, which is related to self-efficacy beliefs. Glanfield (2004) emphasizes that teacher modification is best engendered through professional conversations, which need to occur both within panels (elementary or secondary) and cross-panel (elementary and secondary) to support seamless student transitions across levels. Research will be needed to refine or modify the theory of teaching presented here to better mesh with the realities of the classroom. Davis et al. (2008) point out that promulgating change requires reaching a "critical mass" of educators who embrace the change. Innovation and diffusion research identifies Rogers's *S curve* as a model for adoption of new concepts (Markus, 1987; Rogers, Medina, Rivera, & Wiley, 2005). This theory postulates an S shaped adoption curve. Initially, the innovators adopt the new concept, followed by early adopters. Once a critical mass of adopters is achieved, the rate of adoption increases dramatically, as adopters and then late adopters accept the innovation. After this phase, the rate of innovation acceptance levels off, leaving only the very late adopters and the resistors.

Davis et al. (2008) also point out that real change in teacher behaviour evolves over the teachers' professional lifetime, and is neither instantaneous nor in any sense short term. Professional growth takes significant time; however, formulating and implementing a theory of teaching that is coherent with a theory of learning based on constructivist/complexity, such as Pirie-Kieren, has the potential to improve mathematics learning for students.

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